Yaşadıkça Eğitim, Cilt 38, Sayı 1, Yıl 2024, s.245-257
Journal of Education for Life, Volume 38, Issue 1, Year 2024, pp. 245-257.
DOI: 10.33308/26674874.2024381644

Conceptualization of Out-of-School Mathematics Education

Mehtap KUŞ[1]

Abstract: Out-of-school mathematics education is a relatively new domain. This paper examines learning environments in out-of-school mathematics education, where teachers play a role. These environments fall into three major categories: (a) real-life settings (working places, outdoor environments, daily life settings like home), (b) designed learning environments (e.g., museums, libraries, zoos), and (c) outside classroom settings in schools (e.g., school gymnasiums, art studios). This study highlights interchangeable use of "non-formal" and "informal" education by researchers and raises concerns regarding potential conflicts between informal education programs that employ formal approaches and the inherent nature of informal mathematics education. As a solution, this paper proposes using La Belle's matrix (1982) to analyze current out—of-school mathematics education programs and identify their educational approaches. This contribution will broaden the conceptualization of out-of-school mathematics education and guide future research by examining critical issues.

Keywords: Out-of-School Mathematics Education, Informal Mathematics Education, Out-of-School Learning, Outdoor Mathematics Education, Non-Formal Education

Okul Dışı Matematik Eğitiminin Kavramsallaştırılması

Öz: Okul dışı matematik eğitimi, oldukça yeni ve gelişmekte olan bir alandır. Bu makale, öğretmenin de rol aldığı okul dışı matematik eğitimi ile ilgili öğrenme ortamlarını üç ana kategoride sunmaktadır: (a) gerçek yaşam ortamları (çalışma yerleri, açık hava mekanları, ev gibi günlük yaşam ortamları); (b) tasarlanmış öğrenme ortamları (müzeler, kütüphaneler, hayvanat bahçeleri vb.); (c) okullardaki sınıf dışı ortamlar (okul spor salonu, sanat atölyesi). Çalışmanın bulguları, araştırmacıların formal olmayan (non-formal) ve informal eğitim terimlerini birbiri yerine kullandığını göstermektedir. Ayrıca bu çalışma bazı informal eğitim programlarının formal yaklaşımlar kullandığı ve bu durumun informal matematik eğitiminin doğasıyla çatışabileceği konusundaki endişeleri gündeme getirmektedir. Bu bağlamda, bu çalışma mevcut okul dışı matematik eğitimi programlarını analiz etmek ve kullandıkları yaklaşımları belirlemek üzere La Belle'nin (1982) matrisini kullanmayı önermektedir. Bu çalışma, alandaki kritik konulara dikkat çekerek ve okul dışı matematik eğitiminin daha geniş bir perspektifte kavramsallaştırılmasına katkıda bulunarak gelecekteki araştırmalara ışık tutması ön görülmektedir.

Anahtar Sözcükler: Okul Dışı Matematik Eğitimi, İnformal Matematik Eğitim, Okul Dışı Öğrenme, Açık Havada Eğitim, Formal Olmayan Eğitim

Received: 10.06.2023 Accepted: 28.12.2023 Article Type: Review Article

[1] Aksaray University, Faculty of Education, Department of Mathematics and Science Education, Aksaray, Turkey, e-mail: mozen@aksaray.edu.tr, ORCID: https://orcid.org/0000-0001-7891-9912

To cite:/Atıf için
Kuş, M. (2024). Conceptualization of out-of-school mathematics education. *Yaşadıkça Eğitim*, *38*(1), 245–257. https://doi.org/10.33308/26674874.2024381644

There has been growing interest in out-of-school science learning and free-choice learning, allowing individuals to pursue their personal interests and learn what they want (Falk & Dierking, 2019). In mathematics education, out-of-school learning is a relatively new field compared to science education. However, awareness of its crucial role in various contexts, from museums and outdoor settings to everyday life, is steadily increasing.

The number of museums and research projects on informal mathematics education is growing. Museums worldwide are increasingly offering educational opportunities to engage with mathematics (Nemirovsky et al., 2017). Among these museums with a particular focus on mathematics are The National Museum of Mathematics (MoMath) in New York, The Mathematikum in Germany, Haus Der Mathematik in Austuria, and Thales Museum in Turkey. Additionally, projects such as Project Math-Muse (Guberman et al., 1999), Math in Zoos and Aquariums (MiZA) project (Mokros & Wright, 2009), Math in the Making Project (Rubin & Pattison, 2021), and InforMath (Nemirovsky, 2018) further exemplify this growing interest. The "Mathematics Everywhere" project is another noteworthy example within the Turkish education context (Duatepe-Paksu et al., 2022). Besides research projects, curriculum documents (The report of the Government Office for Science in the United Kingdom (Field & Tuckett, 2016), National Research Council of the United States (2009), Curriculum Report of the Higher Education Council of Turkey (HEC, 2018) highlighted the crucial role of informal learning. In 2018 Curriculum Report of Higher Education Council of Turkey included an elective course in mathematics teacher education program, named "Matematik Öğretiminde Okul Dışı Öğrenme Ortamları" (Mathematics Education in Out-of-School Learning Environments).

There has been growing interest in out-of-school mathematics education, particularly in the context of museum education. However, few studies explicitly describe this field or differentiate it from informal or non-formal education (e.g., Eshach, 2007). Terms like "out-of-school" and "informal" are oftern used interchangeably, or subsumed under "non-formal" education. Nemirovsky and his colleagues (2017) in their comprehensive paper not only characterized informal mathematics education; but also, suggested a fresh perspective on this emerging field. Drawing on prominent studies in mathematics education (Nemirovsky et al., 2017) and broader educational contexts (Eshach, 2007; La Belle, 1982; Rogers et al., 2007), this study aims to present the studies that would contribute to the conceptualization of mathematics education and highlight the critical issues in this emerging field. Furthermore, this study categorizes potential learning environments or contexts for out-of-school mathematics education based on past research. In Turkey, the Higher Education Council (HEC) (2018) included a course on out-of-school mathematics education in teacher training programs. It, however, is not clear how to design the content of the course. Furthermore, there have been very few studies on out-of-school mathematics education in Turkey (e.g., Duatepe-Paksu et al., 2022; Kayhan-Altay & Yetkin Özdemir, 2022). This paper, therefore, aims to assist in curriculum development by explaining the role of out-of-school mathematics education, discussing its conceptualization, and presenting brief examples of learning experiences and environments within this domain.

Theoretical Background

Research in Out-of-School Mathematics Education

Research in out-of-school mathematics education has focused on a wide range of contexts such as museums (e.g., Cooper, 2011; Dancu et al., 2011; Guberman et al., 1999; Gyllenhaal, 2006; Kelton, 2021; Nemirovsky et al., 2013, 2018; Pattison et al., 2012); zoos and aquariums (e.g., Garibay et al. 2012; Mokros & Wright, 2009), outdoor environments (Cahyono & Ludwig, 2018; Fägerstam & Samuelsson, 2014; Moffett, 2011), after-school programs (e.g., Kus & Cakiroglu, 2022; Mueller & Maher, 2009), everyday life settings (e.g., Bonotto, 2005; MacDonald, 2012; Masingila, 1994).

Researchers have shed light on the crucial role of informal learning experiences, exploring visitors' experiences (Cooper, 2011; Guberman et al., 1999; Gyllenhaal, 2006), mathematical thinking and learning in mathematical exhibit visits (Nemirovsky et al., 2013), characterization of informal mathematics education and controversial issues in this emerging field (Nemirovsky et al., 2017), exploration of the ways of connections between informal and formal learning experiences (Kelton, 2021; MacDonald, 2012), discussed the ways of developing current informal learning experiences and design of exhibits (Cooper, 2011; Garibay et al., 2012; Kelton & Ma, 2020; Pattison et al., 2012). These studies used a variety of terms such as outdoor learning, informal mathematics education, and out-of-school mathematics.

In the museum context, Nemirovsky et al. (2017) highlighted mathematical exhibitions as valuable informal learning settings for mathematics education. Museums offer participants open-ended exploration through hands-on, interactive materials, often incorporating technology to create kinesthetic and multisensory experiences. For instance, the "Handling Calculus" exhibition, a collaboration between the Science Museum of Minnesota and TERC, exemplifies this effectively, allowing visitors to grasp calculus concepts through bodily interactions with physical manipulatives (Gyllenhaal, 2006).

Informal settings could be sites for experiencing interdisciplinary learning. They offer opportunities to engage in activities like analyzing data like scientists (Mokros & Wright, 2009), creating mathematical art in studios (Kus & Cakiroglu, 2022), and encountering new perspectives in mathematics (Nemirovsky et al., 2017; 2018). For example, in the study of Kus and Cakiroglu (2022) in an after-school program, students were engaged in studio works that support students' spatial thinking and observed minimalist works of art with a mathematical lens, which involves seeing the works of art differently. Such experiences enable students to appreciate spatial and embodied facets of mathematics, often overshadowed by the numerical and algebraic focus of traditional school settings. Cooper (2011) further emphasizes the significant role of informal settings, suggesting their potential to provide complementary support and enrich formal school mathematics.

Exploration of how informal and formal education can connect is another research focus (Kelton, 2021, MacDonald, 2012). Kelton (2021) highlights the link between school and museum experiences., showing a case of the ways of meaningful link school and museum experiences by connecting conversations between school and museum (e.g., review of vocabulary learned in the classroom, an exhibit becomes the center in the learning of the concept of slope). These museum experiences fostered discovery, curiosity, and playful engagement with mathematics through open-ended materials.

While these studies emphasize the crucial role of out-of-school environments, particularly museums and science centers, some research focuses on improving these settings to enhance mathematical learning. Pattison et al. (2012) investigated the use of computers to guide visitor's interaction with a math exhibit. Their findings suggest that while computers can engage visitors, they may not always help them fully grasp the intended mathematical concepts. Kelton and Ma (2020) suggested participants' embodied activities in the immersive exhibits such as walking would enable them to make sense of the mathematics and exhibition. Besides the design of the exhibits, some researchers drew attention to the role of staff and educators in the exhibits. For example, Pattison et al. (2012) suggested that staff could tailor interactions to families' needs, prior experiences, and visit goals, developing adaptable strategies for various situations. Cooper (2011) suggested ways to develop existing informal settings, such as professional development of docents and educators, connections to curriculum standards, and building family connections through designing brochures as a guide and organizing workshops. In the context of zoo and aquariums, Garibay et al. (2012) highlight the importance of considering institutional size, activity alignment with goals, and staff/educator training.

While researchers have highlighted the significant role of out-of-school mathematics education from various aspects, this emerging field in mathematics education still lacks consensus on standardized terminology. Researchers used a variety of terms, including "outdoor mathematics education", "informal mathematics education", and "everyday mathematics" alongside "out-of-school mathematics education" itself. Very few studies (Nemirovsky et al., 2017) addressed the issue of the scope of informal mathematics education. The next section delves into this topic, drawing upon the work of Eshach (2007), Rogers (2007), and La Belle (1982).

Conceptualization of Out-of-School Mathematics Education

Pattison et al. (2017) reviewed studies focusing on outside of school under two categories: (1) everyday mathematics, encompassing spontaneous learning in daily activities and (2) designed informal mathematics environments such as museums, science centers, and children's museums. In their review of studies on everyday mathematics, they involved studies with a variety of topics ranging from shopping, playing games, budget management, and measuring to construction and nursing. Designed informal learning environments were described as settings where learning occurs with explicit goals of pedagogy and students might have novel experiences of learning mathematics. Designed informal learning environments involve mathematics-themed exhibits in science museums such as MathMoves, Geometry Playground (Dancstep et al., 2015), Handling Calculus (Gyllenhaal, 2006) or mathematics museums such as MoMath (Henebry, 2012), libraries, zoo, or a history museum.

Nemirovsky et al. (2017) introduced the term "informal mathematics education" to distinguish it from the everyday, spontaneous ways we encounter mathematics in daily life. They considered museums to be intentionally designed informal mathematics learning settings dues to their structured schedules, presence of educators, and providing technologies and tools to support mathematical learning. They identified museums, summer camps, clubs, and after-school programs as informal learning settings where students learn mathematics. Beyond characterizing informal mathematics education, they advocated for a new perspective on this field, describing it as a s space where "learners become engaged in questions that matter to them, diversify their sense of what they are capable of, achieve mastery in learning through collaboration, and pursue unanticipated experimentations" (p. 970). This characterization offers valuable insights into this emerging field. Furthermore, they acknowledged the difficulty in clearly differentiating informal and formal education in some cases, while recognizing their stark differences in extreme cases (e.g., silently taking a traditional exam versus freely exploring the exhibits in museums).

Similar arguments have been raised by researchers from other disciplines as well (Eshach, 2007; La Belle, 1982; Rogers, 2007). In the context of science education, Eshach (2007) argued that there is not a clear boundary between formal and informal education and conceptualized informal education differently from the study of Nemirovsky et al. (2017). He divided out-of-school education into two dimensions: non-formal education and informal learning. Non-formal education refers to planned and adaptable educational activities within institutions and organizations, aligning with the concept of designed learning environments within informal mathematics education (Nemirovsky et al., 2017). Informal learning, in contrast, is described as spontaneous learning that occurs in everyday situations. Eshach (2007) identified zoos, science museums, centers, interactive exhibits, and planetariums as non-formal institutions, while classifying homes, parks, streets, and schools as places where learning occurs spontaneously. He also acknowledged that it might not be easy to categorize some institutions as non-formal.

Taking a broader perspective by considering the difference between informal education and informal learning, Rogers (2007) described informal learning as incidental learning, which depends on the participants' needs and desires about what, where, when, and how to learn. Drawing on La Belle's work (1982), he concluded that formal education programs can incorporate non-formal features just as non-formal education programs might include formal elements. This implies that all three forms of learning (formal, non-formal, and informal) can contain the elements of the others. Instead of treating them as separate categories, Rogers conceptualized them as points on a spectrum or continuum ranging from decontextualized (does not change with the different groups of participants) to highly contextualized. Non-formal education encompasses a wide range of activities, including vocational training, continuing educational programs, informal activities within the school, and youth clubs, and extra-curricular activities. He suggested the non-formalization of formal education such as use of extracurricular activities, and the formalization of non-formal education, which means institutionalization of non-formal education to survive to be sustainable. La Belle (1982) proposed a matrix outlining different modes of education (formal, nonformal, and informal education) and their educational characteristics (use of formal, nonformal, and informal approaches). For example, learning in workplace could be an example of informal education; but educators may employ formal methods in this workplace. Another example is that learning within peer group as a part of formal education might occur informally (see further examples of this matrix, La Belle (1982), p. 162). Figure 1 visually represents the ideas presented by La Belle (1982) and Rogers (2007).

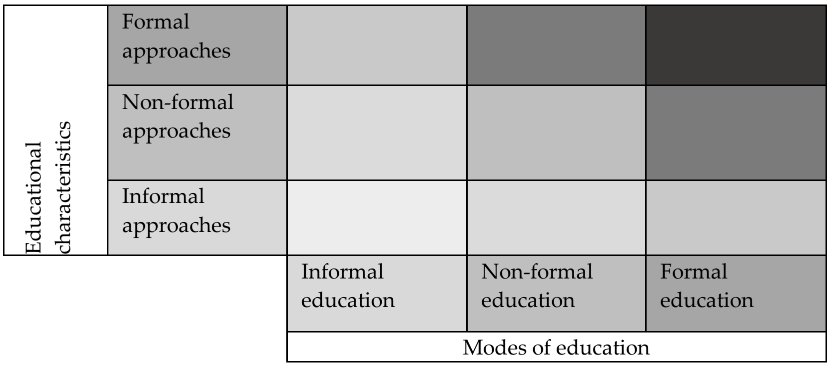


Figure 1. Diagram Merging La Belle's (1992) Matrix of Educational Modes and Their Characteristics with Roger's (2007) Continuum of Modes of Education

The current study does not attempt to define or categorize formal, non-formal, and informal education. Instead, it aims to draw attention to the ongoing discussion surrounding this complex topic. The matrix proposed by La Belle (1982) could be helpful in systematically examining educational characteristics of designed learning environments in mathematics education and being careful about simply generalizing every out-of-school education as happening outside the school or involving informal approaches.

Environments for Out-of-School Mathematics Education

Drawing on previous research, this study examines out-of-school learning environments for mathematics education within three categories: (a) everyday life environments, (b) designed-learning environments, (c) outside classroom activities in schools, and highlights specific examples of out-of-school experiences from the previous studies. In addition to designed learning environments (Nemirovsky et al., 2017) or non-formal learning environments (Eshach, 2007), the current study also includes everyday life environments including outdoor environments where learning takes place with the presence of an educator, excluding spontaneous learning in daily life since the focus of the current study is on teacher education. Please note that there could be numerous out-of-school learning environments other than everyday life environments, designed learning environments such as museums, science centers, and outside classroom activities in schools. The aim of the study is not to comprehensively review all existing research, but rather to organize and describe these environments briefly and provide examples representing education in these environments.

Real Life Environments

While numerous studies have explored the everyday use of mathematics, this study focuses specially on learning opportunities in real-life environments where teachers or educators are present (Lowrie, 2005). Real-life environments in the context of out-of-school mathematics were grouped into three major categories: (1) workplace settings (2) outdoor settings (3) daily-life settings.

Learning occurs in *workplaces* as a part of vocational education or apprenticeships or as an outside school experience. Akkerman and Bakker (2012) suggested establishing continuity between schools and workplaces. There are many examples of the use of mathematics in workplaces highlighted by several researchers (proportional reasoning in nursing practice, Hoyles et al., 2001; use of mathematics in carpet lying, Masingila, 1994; Wake, 2014). Technicians in laboratory settings often have competencies such as dealing with and interpreting data sets, measurement, using and creating mathematical diagrams, and proportional reasoning (Wake, 2014). For example, in the workplace visits college students had opportunities to observe and experience how a railway signal engineer work and use mathematics such as calculating the speed of trains and the distance between points of stop and warning boards of speed (Wake, 2014). Wake drew attention to the importance of preparing students as workers in workplace settings and expanding the usual conception of schools by encouraging students to have everyday life experiences. Beyond workplace visits, Masingila (1994) described how carpet layers use mathematics and identified mathematical processes that carpet layers made use of such as measuring, estimating, visual-spatial arrangement of layers, problem-solving, using tools, and algorithms. Masingila (1993) further underscored the importance of practicing mathematics in context, bridging the gap between school and everyday life. This led to the suggestion of employing an apprenticeship model within classroom settings to create more authentic learning experiences. On the other hand, this issue reminds the concerns raised by Nemirovsky et al. (2017) about whether creating simulated workplace contexts within schools such as a shopping scenario can truly qualify as be authentic informal mathematics education.

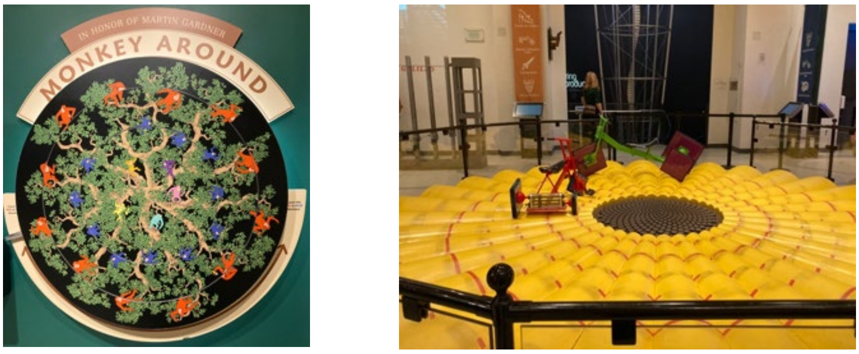
Outdoor environments refer to the places like parks, playgrounds, streets, and forests in the current study. Moss (2009) referred to learning and teaching mathematics in outdoor places as "outdoor mathematics education". Moss (2009) described outdoor mathematics activities as activities that help students experience and explore the world outside, experience practicing mathematics in a larger place that is not provided in school settings, and do mathematics outside in an enjoyable manner in a setting different from the school environment. Moss proposed engaging activities like exploring geometric shapes in nature, mathematical patterns on sidewalks, embarking on orienteering adventures. There have been studies that could be considered within the scope of outdoor mathematics education (e.g., Cahyono & Ludwig, 2018; Haas et al., 2021; Watson et al., 2011). The activity proposed by Watson and her colleagues (2011) which included measuring the height of a tree using a clinometer and conducting statistical analysis could be considered as an example of outdoor mathematics education. Cahyono and Ludwig (2018) suggested another activity that can be implemented outside of the classroom by using digital technology. Students were asked to solve math trail tasks scattered around the city by using a mobile application and map. For example, in one of the tasks students were asked to estimate the base area of a historical building which require students to identify the geometric shape of the base and measure its dimensions to calculate its area.

While extensive research exists on *everyday mathematics*, this study focuses specifically on bridging the gap between informal and formal mathematics in guided learning settings led by a teach, rather than unplanned experiences. Daily life locations like homes, markets, shopping centers, cinemas, and theatres serve as the context for these explorations. In one of the studies suiting well within this description, MacDonald (2012) used photographs as a way to explore children's experiences (5-6 years old) at home with measurement and as a way to connect home and school activities. Photo elicitation was used as a research method, which was used as a catalyst for further discussion. Children were asked to take photos at their homes that can be related to measurement concepts such as pictures of utensils used to measure while cooking, and a picture of their showing the difference in height between the child and his grandmother. The author reported that the use of photographs gave students opportunities to have personal and meaningful connections with mathematics and enabled connections between school and home (in school and out-of-school environments). In another study, Bonotto (2005) examined the ways of connecting informal mathematics and formal mathematics, particularly in the context of the multiplication of decimal numbers. The author used supermarket receipts as a way of connecting informal and formal mathematics since they include the weight of the items represented in decimal numbers. There could be limited studies within this category because creating situations such as shopping context in the classroom (Brenner, 1998) may not be authentic within the scope of informal mathematics education, or visiting such places with a large group of students might not always be very feasible.

Designed Learning Environments

Designed learning environments refer to informal settings intentionally designed because of their schedules, having educators, and providing technologies and tools to support the learning of mathematics (Nemirovsky et al., 2017).

Mathematics museums and exhibits, like MoMath in New York (Henebry, 2012; Lawrence, 2023) or Mathematikum in Germany (Beutelspacher, 2018), offer unique spaces for students to experiment with mathematical ideas and explore their connections to other disciplines, like science. Beutelspacher (2012) even calls them "mathematical science centers." In recent years, the number of such exhibits has been steadily increasing (Beutelspacher, 2018). MoMath found in New York is one of the popular museums that is particularly centered on mathematics and opened in 2012. MoMath has a wide range of collections such as square-wheeled tricycle, coaster roller, hyper hyperboloid, and monkey around (Figure 2). For example, in the exhibit of Hyper Hyperboloid visitors understand straight lines can form a curved surface. Another prominent material is square-wheeled tricycles. With this material, visitors explore how a square-wheeled tricycle rides smoothly and the mathematics behind it (use of a catenary curve) (Rosmarin, 2015). Rosmarin (2015) also reported youths' experiences in museums. The museum enables us to experience visual mathematics rather than numbers and algebra, physical interaction with mathematics, and see diverse images of mathematics, which are not provided in school mathematics and provide new ways of mathematics thinking and learning.



(a) (b)

Figure 2. Examples of Materials at MoMath (a) Monkey Around, (b) Square-Wheeled Tricycle (Photos taken by the author on 22 September 2022)

Science museums and similar institutions (e.g., children's museums) refer to the places where students could explore scientific and mathematical ideas (e.g., Kelton, 2015; Nemirovsky et al., 2013; Vandermaas-Peeler et al., 2016). Science museums offer mathematically oriented exhibits with an increasing number (Nemirovsky et al., 2013); mathematics-themed exhibits in science museums such as MathMoves, Geometry Playground in Exploratorium Science Museum (Dancstep et al., 2015; Dancu et al., 2011), Handling Calculus in Science Museum of Minnesota (Gyllenhaal, 2006); MadeinMath in Science Museum of Trento (Andreatta, 2018). Similarly, there are some mathematics-themed exhibits in Science Centers and Museums in Turkey such as Konya Science Center (Figure 3). Research has delved into visitors' experiences with these exhibits. Gyllenhaal (2006) explored visitors' experiences in an interactive mathematics-themed exhibit called Handling Calculus. The exhibit enabled visitors to explore motion graphs and differentiation and integration concepts, slope concept, and limit. Dancstep et al. (2015) explored visitors' experiences with the mathematics-oriented exhibit in the science center called Geometry Playground developed by Exploratorium, a science museum in San Francisco. They compared the visitors' (adult and children) experiences in immersive exhibits, where visitors can go inside or surround, and table-top exhibits, where visitors can manipulate their hands. They compared three exhibits named, Voronoi diagrams, and transformative and rotational solid exhibits. Andreatta (2018) reported that MadeinMath visitors found mathematics appealing and interactive rather than seeing the abstract nature of mathematics. Bennett (2020) explored the ways of visitors engage in the mathematics exhibit as an informal mathematics learning environment in a science center developed in the partnership with Mathematics Department of University of Arizona. The exhibitions included puzzles and games related to mathematics. The author investigated three of twenty puzzles. The puzzles focused on logical reasoning and addition concept. Each puzzle asked visitors to add numbers so that the sum of them has a common reaches a particular value (e.g., 1+13= 14, 3+11=14, p. 25) or reaches less than a particular number.

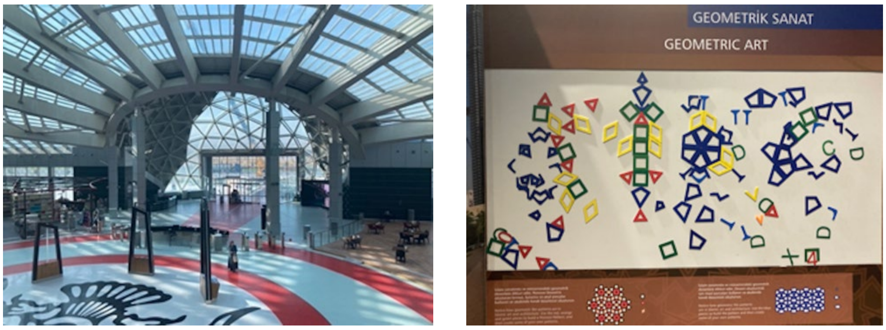
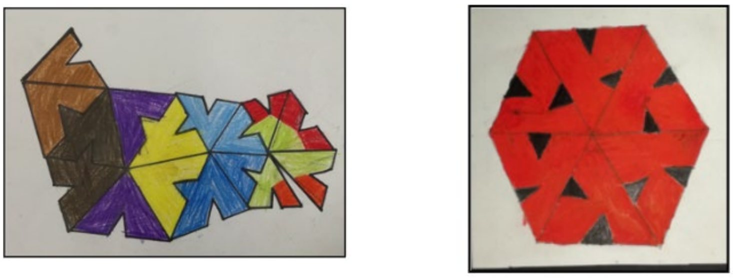


Figure 3. A Math-themed Exhibit, *Islamic Geometric Art* from Konya Science Center in Turkey (Photos taken by the author on 8 November 2020)

History, art museums and galleries, and other environments such as zoos, aquariums, libraries, and natural history museums also fall into the scope of out-of-school mathematics education. Art museums, both in Turkey (e.g., Pera Museum, İstanbul Modern Museum, CerModern) and abroad (e.g., Museum of Modern Art, National Gallery of in the U.S., Escher in the Palace in the Netherlands), are increasingly organizing educational programs for children and adults. Fathauer (2007) reported that the number of mathematical art exhibitions is increasing over the last ten years. Bridges Conference, Joint Mathematics Meetings (Fathauer, 2007), and mathematical art exhibitions in MoMath (Lawrence, 2023) are indicators of such an increase. While research in this area remains limited (e.g., Barry & Villeneuve, 1998; Neu, 1985; Nemirovsky, 2018), existing studies offer promising insights. Neu (1985), for example, put importance on the role of the art museum in learning disciplines other than arts in the context of the Delaware Art Museum. For example, students observed sculptures based on geometric principles and geometric shapes and were asked to create their sculptures in the museum's studio. Nemirovsky and his colleagues (2018), in the context of a basket weaving program in an after-school club, examined students' engagement with museum materials such as baskets exhibited in an art museum. They put the importance of emerging learning and the role of perception and imagination during the analysis of baskets. Basket context encouraged them to explore ideas about the curvature of surfaces. There are also two studies in the context of zoos and aquariums (Garibay et al., 2012; Mokros & Wright, 2009). Mokros and Wright (2009), for example, presented an activity that enables children to deal with scientific observational data including collecting and analyzing data, representing and interpreting, and making predictions from the data in the informal settings of zoos and aquariums by exploring the patterns in animals' behaviors.

Outside Classroom Settings (e.g., Schoolyard, School Gymnasium, Art Studios)

Eshach (2007) challenged the narrow view of non-formal education as solely occurring outside school. He argued that learning extends beyond the classroom walls, encompassing spaces within the school environment like the schoolyard, gymnasium, art studios, and technology centers. For example, Kus and Cakiroglu's (2022) study within an art-math after-school program explored students' spatial thinking in a public-school art studio. Studio works involving artmaking enable students to explain their ideas freely and create their artworks by making use of mathematics. For example, students created their artworks by inspiring from artworks (V series) of Frank Stella (Figure 4). The study of Kelton and Ma (2018) could be interpreted within this category even though researchers might not particularly aim to conduct this study within the scope of out-of-school mathematics education. They investigated the role of students' whole-body collaboration in making sense of mathematical ideas regarding number sense, ratio, and proportion in the school's gymnasium where students attended a series of activities about number line on which students can walk, and in math class. In the schoolyard context, Vela et al. (2022) presented an activity where students engaged with measurement (perimeter and area calculations) and spatial reasoning (scale drawing and map creation) to design a school garden. Students acting like architects, engineers or designers had to decide the location, size, and layout of the garden.



(a) (b)

Figure 4. Students' Artworks Inspired by Frank Stella's V Series in Math-Art Studio (Kus, 2019)

Conclusion and Future Directions

Drawing on the previous research, this study presented brief examples of diverse out-of-school learning environments and categorized them into three main types: (a) everyday life settings, (b) designed learning environments, and (c) outside classroom settings in schools. It is noteworthy that these contexts were included in the teacher education course "Matematik Öğretiminde Okul Dışı Öğrenme Ortamları" (Mathematics Education in Out-of-School Learning Environments), implemented by the Higher Education Council of Turkey in 2018. The aim of the study was not to cover all existing research within the context of out-of-school mathematics education, but rather briefly categorize and explain these environments, drawing on examples from past research. In addition, this study raises critical issues about the conceptualization and scope of out-of-school mathematics education by mainly taking into consideration the studies (Eshach, 2007; La Belle, 1982; Nemirovsky et al., 2017; Rogers, 2007), what kind of learning opportunities or environments could be considered within the scope of out-of-school mathematics education.

One of the key concerns identified in the literature is the characterization of informal or out-of-school mathematics education. Since there is scarce research in informal mathematics education, it is hard to find a consensus on this matter. Researchers also used a variety of terms regarding designed learning settings such as museums, zoos, libraries, and science centers (informal learning setting (Nemirovsky et al., 2017; Pattison et al., 2017); non-formal education settings (Eshach, 2007). Nemirovsky et al. (2017)'s focus on designed learning environments excludes spontaneous everyday math learning from informal education. They also propose that distinguishing informal and formal education can be challenging in some cases, echoing similar arguments from La Belle (1982) and Rogers (2007). Instead of seeking rigid boundaries, education can be conceptualized as a continuum (from planned compulsory education to unplanned and incidental learning) and explored through La Belle's (1982) three-dimensional matrix, which argues that formal, non-formal, and informal education may coexist, sometimes collaborating and sometimes conflicting. For example, Nemirovsky et al. (2017) discussed that museum as one of the major settings for informal mathematics education would have pressure to offer educational activities in line with curriculum objectives in formal mathematics education. The matrix proposed by La Belle (1982) would be helpful to examine the current programs, activities, or museum-based experiences and revise them depending on the purpose of the institutions or educators. The aim of the current study is not to solve the controversial issue about the characterization of out-of-school mathematics education and discriminate it from formal education. Rather, the purpose is to bring up these issues to the readers' attention by taking into consideration the studies in other fields, describing informal, non-formal education, and formal education, and adapting them to the context of mathematics education. These studies could be helpful to see its relation with other forms of education in a broad sense and to identify what kind of approaches are used by the staff facilitator, educators, and teachers.

This study reviewed the studies related to out-of-school mathematics education to examine possible contexts for out-of-school mathematics education. These learning environments were categorized into three major categories: (a) everyday contexts like professional and natural environments where teachers or educators are present; (2) designed learning environments like science museums and centers. It is noteworthy that the focus of the current study is on the studies in which teacher has a role in visiting or incorporating them into school mathematics and thus, it did not include studies on learning that happen in spontaneous and unplanned ways. Furthermore, the categorization of Eshach (2007) was adapted to the context of the study. In addition to designed (Nemirovsky et al., 2017) or non-formal learning environments (Eshach, 2007), the current study also includes everyday life environments including outdoor environments where learning takes place with the presence of an educator and extra-curricular activities in a school environment, excluding spontaneous learning in daily life. Eshach (2007) argued that the distinction between different modes of education is not simply about the physical differences (whether it is in or outside of the school), rather it is about other elements such as motivation and interest of visitors or students, social context of learning, and assessment. While this study includes studies on everyday life settings, this study also reminds two dilemmas raised by Nemirovsky et al. (2017). One is that some everyday contexts (creating a context for shopping) might not be authentic and efforts for connecting mathematics with real-life sometimes might be forced. Considering these issues, and acknowledging that not all contexts or concepts are suitable for out-of-school mathematics education, this study asks critical questions if there are authentic contexts regarding everyday mathematics within the scope of informal mathematics education or if authentic outside classroom learning activities could be considered within the scope of out-of-school mathematics education.

Despite limited research on characterizing and comparing mathematical learning in different out-of-school environments, a review of existing studies revealed some common aspects regarding mathematics education in these learning settings. Researchers emphasized the values of personal experience in programs and activities like creating mathematical artwork in art studios (Kus & Cakiroglu, 2022) or collecting data in zoos and aquarium visits (Mokros & Wright, 2009). These studies drew attention to hands-on-experience and seeing diverse ways of mathematics rather than just focusing on numbers and algebra. Particularly studies conducted in the art-related environments and museum settings encouraged free choice opportunities in mathematics learning such as creating their own artworks or interacting with materials that they find as interesting. This highlights the need for future research to delve into the nature of mathematical thinking and learning in such environments, exploring the structure and level of guidance offered by various activities and materials. Furthermore, comparing the learning opportunities provided by different mathematics museums and science centers based on visitor experiences is crucial. Such research would enrich our understanding of out-of-school mathematics education and inform educational policies in this area. Ultimately, this line of inquiry could foster stronger partnerships between schools and museums, leading to the development of existing institutions and the creation of new ones.

In conclusion, this review informs readers what kind of issues, dilemmas, and controversial issues are discussed within the scope of out-of-school education and brought up the question of what counts as informal mathematics education to the attention of the readers. This study also provides readers with brief examples of out-of-school learning environments and presents the crucial role of out-of-school mathematics education. This study should be interpreted within the limitations of the study. This does not provide a systematic review of the literature or cover all informal educational environments for out-of-school education. This study offers new insights by interpreting the works of Eshach (2007), Roger (2007), and La Belle (1982) through a mathematics education lens. The study informs the design of teacher education courses, raising awareness and broadening understanding of this area among educators. Future research would shed light on the effectiveness of current programs through proposed matrix and explore development strategies aligned with the nature of informal mathematics education described by Nemirovsky et al. (2017).

Declarations

Acknowledgements: Not applicable.

Authors' contributions: Responsible author contributed to the manuscript.

Competing interests: The author declare that they have no competing interests.

Funding: The author received no financial support for the study.

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